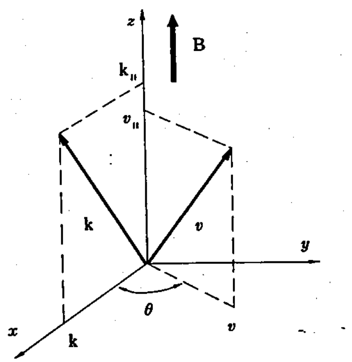

Fig. 3.1 Configuration of various vectors.

In carrying out those integrations, we specifically choose the wave vector **k** on the x–z plane (see Fig. 3.1) and write

$$\mathbf{k}=k_\perp\hat{x}+k_\parallel\hat{z}. \tag{3.67}$$

We also note that

$$\frac{\partial f_\sigma(\mathbf{v}')}{\partial \mathbf{v}'}=\frac{\partial f_\sigma}{\partial v_\perp}\left[\cos(\Omega_\sigma\tau+\theta)\hat{x}+\sin(\Omega_\sigma\tau+\theta)\hat{y}\right]+\frac{\partial f_\sigma}{\partial v_\parallel}\hat{z}. \tag{3.68}$$

Since

$$\phi(\tau)=\mathfrak{z}\left[\sin(\Omega_\sigma\tau+\theta)-\sin\theta\right]+k_\parallel v_\parallel\tau-\omega\tau$$

where

$$\mathfrak{z}\equiv k_\perp v_\perp/\Omega_\sigma, \tag{3.69}$$

we have†

$$\exp[-i\phi(\tau)-\eta\tau]=\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}J_n(\mathfrak{z})J_{n'}(\mathfrak{z})$$
$$\times\exp\{-i[n(\Omega_\sigma\tau+\theta)-n'\theta+k_\parallel v_\parallel\tau-\omega\tau]-\eta\tau\}. \tag{3.70}$$

We substitute (3.64), (3.65), (3.68), and (3.70) in Eq. (3.66); integration with respect to θ then yields selection rules for the integers n and n' in the summations of Eq. (3.70). After a substantial amount of algebraic manipula-

† Note that $\exp(-iz\sin\psi)=\sum_{n=-\infty}^{\infty}J_n(z)\exp(-in\psi)$ where J_n is the Bessel function of the nth order.

tion, which we leave to the reader as an exercise (Problem 3.7), we find

$$\epsilon(\mathbf{k},\omega)=\left(1-\frac{\omega_p^2}{\omega^2}\right)\mathbf{I}-\sum_\sigma\frac{\omega_\sigma^2}{\omega^2}\sum_{n=-\infty}^{\infty}\int d\mathbf{v}\left(\frac{n\Omega_\sigma}{v_\perp}\frac{\partial f_\sigma}{\partial v_\perp}+k_\parallel\frac{\partial f_\sigma}{\partial v_\parallel}\right)$$
$$\times\frac{\Pi_\sigma(v_\perp,v_\parallel;n)}{n\Omega_\sigma+k_\parallel v_\parallel-\omega-i\eta} \tag{3.71}$$

where

$$\Pi_\sigma(v_\perp,v_\parallel;n)=\begin{bmatrix}\dfrac{n^2\Omega_\sigma^2}{k_\perp^2}J_n^2 & iv_\perp\dfrac{n\Omega_\sigma}{k_\perp}J_nJ'_n & v_\parallel\dfrac{n\Omega_\sigma}{k_\perp}J_n^2\\ -iv_\perp\dfrac{n\Omega_\sigma}{k_\perp}J_nJ'_n & v_\perp^2(J'_n)^2 & -iv_\parallel v_\perp J_nJ'_n\\ v_\parallel\dfrac{n\Omega_\sigma}{k_\perp}J_n^2 & iv_\parallel v_\perp J_nJ'_n & v_\parallel^2J_n^2\end{bmatrix}, \tag{3.72}$$

$$\int d\mathbf{v}\equiv2\pi\int_0^\infty v_\perp dv_\perp\int_{-\infty}^{\infty}dv_\parallel,\qquad J_n=J_n(\mathfrak{z}),\qquad J'_n=\frac{dJ_n(\mathfrak{z})}{d\mathfrak{z}}, \tag{3.73}$$

and $\mathfrak{z}$ has been defined by (3.69). We remark in passing that the last term in Eq. (3.71) decreases as ω^{-3} in the limit of very high frequencies; the calculation Eq. (3.71) is therefore in accord with general asymptotic behavior, Eq. (3.49).

C. The Dielectric Response Function

With the aid of (3.36) again, we may calculate from (3.71) an explicit expression for the dielectric response function in a magnetic field; the result is

$$\epsilon(\mathbf{k},\omega)=1-\sum_\sigma\frac{\omega_\sigma^2}{k^2}\sum_{n=-\infty}^{\infty}\int d\mathbf{v}\left(\frac{n\Omega_\sigma}{v_\perp}\frac{\partial f_\sigma}{\partial v_\perp}+k_\parallel\frac{\partial f_\sigma}{\partial v_\parallel}\right)\frac{J_n^2(\mathfrak{z})}{n\Omega_\sigma+k_\parallel v_\parallel-\omega-i\eta}. \tag{3.74}$$

This function is useful in describing the longitudinal fluctuations in the magnetic field.

with the aid of (5.10), we find

$$E_y(x) = -i\frac{\omega}{\pi c}B(+0)\int_{-\infty}^{\infty} dk_x \frac{\exp(ik_x x)}{k_x^2 - i(4\pi\omega/c^2)\sigma_T(|k_x|,\omega)}. \qquad (5.15)$$

The anomalous skin effect is usually measured through the surface impedance, which is calculated from (5.15) as

$$\mathcal{Z} \equiv (E_y/B_z)_{x=+0}$$

$$= -i(\omega/\pi c)\int_{-\infty}^{\infty} dk_x\left[k_x^2 - i(4\pi\omega/c^2)\sigma_T(|k_x|,\omega)\right]^{-1}. \qquad (5.16)$$

As may be clear from the derivation above, the applicability of Eqs. (5.15) and (5.16) is not restricted to cases of classical plasmas. With the knowledge of the transverse conductivity, we can use these formulas for any conductive medium.

For the classical electron gas, the conductivity is given by (5.7). Assuming (5.8), we may approximate the exponential function in (5.7) to be unity. Substitution of this approximate expression into Eq. (5.16) then yields

$$\mathcal{Z} = \frac{2\omega\delta}{3c}\left(\frac{1}{\sqrt{3}} - i\right) \qquad (5.17)$$

where

$$\delta = \left[\left(\frac{2T}{\pi m}\right)^{1/2}\frac{c^2}{\omega_p^2\omega}\right]^{1/3} \qquad (5.18)$$

is the anomalous skin depth of the classical plasma (Problem 5.3).

In the light of this specific calculation, let us finally examine the validity of the assumption on (5.8). Within the skin layer, $k \gtrsim 1/\delta$; hence,

$$\frac{\omega}{k(T/m)^{1/2}} \lesssim \frac{\omega\delta}{(T/m)^{1/2}} = \left[\left(\frac{2}{\pi}\right)^{1/2}\frac{\omega^2c^2}{\omega_p^2(T/m)}\right]^{1/3}. \qquad (5.19)$$

The finiteness of a laboratory plasma places a lower bound on the frequencies usable in such a skin-effect experiment. For those parameters available to classical plasmas, it then appears quite difficult to make the right-hand side of (5.19) less than unity.

If, however, we move over to the cases of degenerate plasmas in solids, the situation changes significantly. Here, we may approximately replace T by the Fermi energy in expressions such as (5.18) and (5.19). In addition, the plasma frequency increases substantially because of high concentration and difference in the effective mass. Experimentally, therefore, the anomalous skin effect is more easily observed in a solid state plasma than in a gaseous plasma.

5.2 DIELECTRIC TENSOR WITH AN EXTERNAL MAGNETIC FIELD

We proceed to consider the cases of a plasma with a uniform stationary magnetic field applied to it. The system is no longer isotropic; for the analysis of electromagnetic wave propagation in such a system, all the elements of the dielectric tensor must be calculated.

The velocity distribution function under present consideration is the Maxwellian, (4.66). The explicit expression for the dielectric tensor may be obtained by substituting this distribution function into Eq. (3.71). The intermediate steps of the calculation are left to the reader as an exercise (Problem 5.4); the result for a single-component plasma is

$$\boldsymbol{\epsilon}(\mathbf{k},\omega) = \mathbf{I} - \frac{\omega_p^2}{\omega^2}\left\{\sum_{n=-\infty}^{\infty}\frac{Z_0}{Z_n}\boldsymbol{\Pi}(\beta,Z_n;n)\left[1 - W(Z_n)\right] - Z_0^2\hat{z}\hat{z}\right\} \qquad (5.20)$$

where

$$\boldsymbol{\Pi}(\beta,Z_n;n) = \begin{bmatrix} \frac{n^2}{\beta}\Lambda_n(\beta) & in\Lambda_n'(\beta) & \frac{k_\parallel}{|k_\parallel|}\frac{n}{\sqrt{\beta}}Z_n\Lambda_n(\beta) \\ -in\Lambda_n'(\beta) & \frac{n^2}{\beta}\Lambda_n(\beta) - 2\beta\Lambda_n'(\beta) & -i\frac{k_\parallel}{|k_\parallel|}\sqrt{\beta}\,Z_n\Lambda_n'(\beta) \\ \frac{k_\parallel}{|k_\parallel|}\frac{n}{\sqrt{\beta}}Z_n\Lambda_n(\beta) & i\frac{k_\parallel}{|k_\parallel|}\sqrt{\beta}\,Z_n\Lambda_n'(\beta) & Z_n^2\Lambda_n(\beta) \end{bmatrix}, \qquad (5.21)$$

$$Z_n = \frac{\omega - n\Omega}{|k_\parallel|(T/m)^{1/2}}. \qquad (5.22)$$

Equations (4.2), (4.70), and (4.69) define the functions W and Λ_n and the variable β, respectively; the configuration as specified in Fig. 3.1 has been used for the vectors **B** and **k**. Equation (5.20) is extended to the case of a multicomponent plasma in a standard way.

and the equation of motion

$$n_\sigma\left(\frac{\partial}{\partial t}+\mathbf{u}_\sigma\cdot\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{u}_\sigma=-\gamma_\sigma\frac{T_\sigma}{m_\sigma}\frac{\partial n_\sigma}{\partial \mathbf{r}}+\frac{q_\sigma}{m_\sigma}n_\sigma\left(\mathbf{E}+\frac{\mathbf{u}_\sigma}{c}\times\mathbf{B}\right)-\frac{n_\sigma\mathbf{u}_\sigma}{\tau_\sigma}.$$

Here, $\mathbf{u}$ is the flow velocity, τ the relaxation time, and γ a constant of order unity. For the unperturbed state, we take a homogeneous plasma with a constant uniform magnetic field $\mathbf{B}$ in the z direction; we assume no net flows of the particles (i.e., $\mathbf{u}_\sigma=0$) in this state. Calculate the dielectric tensor for this system and show that it may be expressed as

$$\boldsymbol{\epsilon}(\mathbf{k},\omega)=\mathbf{I}+\sum_\sigma\frac{\omega_\sigma^2}{\omega^2(1-i\omega\tau_\sigma)}\frac{i\tau_\sigma\omega\mathbf{b}}{\omega(1-i\omega\tau_\sigma)+i(\gamma_\sigma T_\sigma\tau_\sigma/m_\sigma)\mathbf{k}\cdot\mathbf{b}\cdot\mathbf{k}}$$

$$\cdot\left\{\left[\omega(1-i\omega\tau_\sigma)+i\frac{\gamma_\sigma T_\sigma\tau_\sigma}{m_\sigma}\mathbf{k}\cdot\mathbf{b}\cdot\mathbf{k}\right]\mathbf{I}-i\frac{\gamma_\sigma T_\sigma\tau_\sigma}{m_\sigma}\mathbf{kk}\cdot\mathbf{b}\right\}$$

where

$$\mathbf{b}=\left[(1-i\omega\tau_\sigma)^2+(\Omega_\sigma\tau_\sigma)^2\right]^{-1}$$

$$\times\begin{bmatrix}(1-i\omega\tau_\sigma)^2 & \Omega_\sigma\tau_\sigma(1-i\omega\tau_\sigma) & 0\\ -\Omega_\sigma\tau_\sigma(1-i\omega\tau_\sigma) & (1-i\omega\tau_\sigma)^2 & 0\\ 0 & 0 & (1-i\omega\tau_\sigma)^2+(\Omega_\sigma\tau_\sigma)^2\end{bmatrix}.$$

Check the asymptotic property (3.49) for this dielectric tensor.

3.6. Derive Eq. (3.50).

3.7. Carry out calculations leading from Eq. (3.66) to Eq. (3.71). The following formulas may be used in this derivation:

$$(2n/\mathfrak{z})J_n(\mathfrak{z})=J_{n-1}(\mathfrak{z})+J_{n+1}(\mathfrak{z}),\qquad 2J_n'(\mathfrak{z})=J_{n-1}(\mathfrak{z})-J_{n+1}(\mathfrak{z}),$$

$$\sum_{n=-\infty}^{\infty}J_n^2(\mathfrak{z})=1,\qquad \sum_{n=-\infty}^{\infty}[nJ_n(\mathfrak{z})]^2=\mathfrak{z}^2/2.$$

3.8. Derive Eq. (3.74).

3.9. Show directly from (3.74) that this function has the asymptotic property (3.51).

CHAPTER 4

LONGITUDINAL PROPERTIES OF A PLASMA IN THERMODYNAMIC EQUILIBRIUM

In Chapters 2 and 3, we have studied the basic theoretical tools for investigating various plasma phenomena. In particular, we have learned that the dielectric tensor is capable of describing both the longitudinal and transverse properties of a plasma. In this chapter, we are concerned with the longitudinal properties of a plasma in thermodynamic equilibrium. We shall, therefore, be dealing with the dielectric response function evaluated with a Maxwellian velocity distribution; a function of a complex variable, the plasma dispersion function, will be introduced in this context.

Our first topic of study will be the properties of the collective modes in such a plasma; the plasma oscillation and its damping will be discussed. We shall next turn to those phenomena related to the screening properties of the dielectric response function: these include the Debye screening, the stopping power of the plasma, and the Cherenkov emission of the plasma oscillations. We shall then extend our consideration to those cases of a two-component plasma in which the positive ions as well as the electrons participate in the dynamical processes. In such a system, we shall find that a new mode of wave propagation, the ion-acoustic wave, is possible in addition to the ordinary plasma oscillation; the properties of the ion-acoustic wave will be investigated. Finally, we shall consider the cases in which a uniform magnetic field is applied to the plasma. The propagation characteristics of both the plasma oscillation and the ion-acoustic wave will then be modified; an additional mode of propagation associated with the cyclotron motion of the charged particles will also appear. The latter is called the Bernstein mode.

4.1 THE PLASMA DISPERSION FUNCTION

Since we are concerned here with a plasma in thermodynamic equilibrium, we evaluate the dielectric response function with the aid of a Maxwellian velocity distribution (1.24). The system is isotropic; we may arbitrarily choose the x axis in the direction of the wave vector. For the electron-gas model of the

5.4. Carry out the calculations leading to Eq. (5.20). In the derivation, it may be helpful to note

$$\sum_{n=-\infty}^{\infty} \pi(v_\perp, v_\parallel; n) = \begin{bmatrix} v_\perp^2/2 & 0 & 0 \\ 0 & v_\perp^2/2 & 0 \\ 0 & 0 & v_\parallel^2 \end{bmatrix},$$

which is derived with the aid of the formulas in Problem 3.7; and

$$\int_0^\infty dx\, x^2 J_n(x) J_n'(x) \exp\left(-\frac{x^2}{2\beta}\right) = \beta^2 \Lambda_n'(\beta),$$

$$\int_0^\infty dx\, x^3 [J_n'(x)]^2 \exp\left(-\frac{x^2}{2\beta}\right) = n^2\beta\Lambda_n(\beta) - 2\beta^3\Lambda_n'(\beta),$$

which obtain from differentiations of Eq. (4.67) with respect to p, q, or both.

5.5. When the dielectric tensor is expressed as

$$\boldsymbol{\epsilon}(\mathbf{k},\omega) = \begin{bmatrix} \epsilon_1 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}$$

in the Cartesian coordinate system, show that the dispersion relation Eq. (5.23) becomes

$$[\epsilon_1 \sin^2\theta + \epsilon_3\cos^2\theta](kc/\omega)^4 - [(\epsilon_1+\epsilon_2)(\epsilon_1-\epsilon_2)\sin^2\theta + \epsilon_1\epsilon_3(1+\cos^2\theta)](kc/\omega)^2 + (\epsilon_1+\epsilon_2)(\epsilon_1-\epsilon_2)\epsilon_3 = 0,$$

where θ is the angle between $\mathbf{k}$ and the z axis. This equation may then be solved for $(kc/\omega)^2$ as

$$\left(\frac{kc}{\omega}\right)^2 = \frac{[(\epsilon_1+\epsilon_2)(\epsilon_1-\epsilon_2)\sin^2\theta + \epsilon_1\epsilon_3(1+\cos^2\theta)] \pm \{[(\epsilon_1+\epsilon_2)(\epsilon_1-\epsilon_2) - \epsilon_1\epsilon_3]^2 \sin^4\theta + 4\epsilon_2^2\epsilon_3^2\cos^2\theta\}^{1/2}}{2[\epsilon_1\sin^2\theta + \epsilon_3\cos^2\theta]},$$

or for θ as

$$\tan^2\theta = -\frac{\epsilon_3[(kc/\omega)^2 - (\epsilon_1+\epsilon_2)][(kc/\omega)^2 - (\epsilon_1-\epsilon_2)]}{[(kc/\omega)^2\epsilon_1 - (\epsilon_1+\epsilon_2)(\epsilon_1-\epsilon_2)][(kc/\omega)^2 - \epsilon_3]}.$$

5.6. Consider the effects of particle collisions on the low-frequency propagation modes of the electromagnetic wave in plasmas along the magnetic field, using the expression for the dielectric tensor obtained in Problem 3.5; we may assume $T_\sigma = 0$ in that expression by neglecting the thermal effects.

(a) Show that the helicon mode now suffers attenuation due to the collisions as described by

$$\omega = \frac{|\Omega_e|c^2}{\omega_e^2}k^2\left(1 - \frac{i}{\tau_e|\Omega_e|}\right).$$

(b) Show that for the Alfvén wave the dispersion relation becomes

$$\omega = c_A k - \frac{i}{2(m_e+m_i)}\left(\frac{m_e}{\tau_e} + \frac{m_i}{\tau_i}\right).$$

5.7. Derive Eqs. (5.73)–(5.76).

5.8. Consider propagation of an electromagnetic wave in a compensated plasma consisting of electrons and singly charged ions; the effects of the thermal motion and of collisions may be neglected. The frequency of the wave is fixed at a value twice the plasma frequency [i.e., $\omega = 2(\omega_e^2 + \omega_i^2)^{1/2}$]. Assuming (4.45), we take account of only those terms up to the first order in (m_e/m_i). The direction of the magnetic field is either parallel or perpendicular to the propagation direction of the electromagnetic wave. Only the strength B of the magnetic field is gradually increased from $B=0$ to a value that satisfies $(eB/m_i c\omega)^2 \gg 1$. Discuss the major changes in the propagation characteristics that may take place during these processes.