Waves in Plasma

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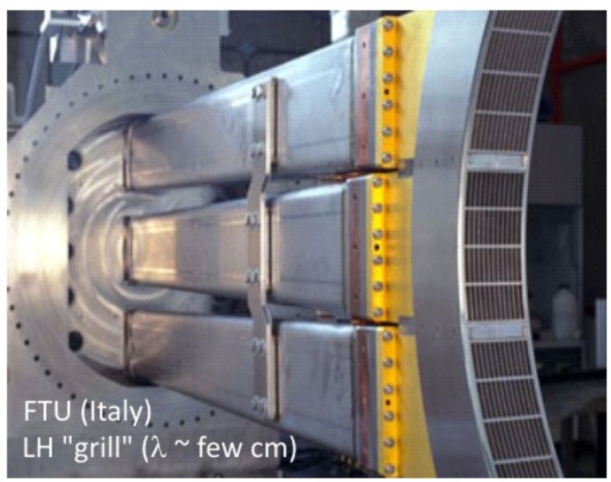
- General dispersion of linear waves
- Waves in homogeneous stationary plasmas
- Wave evolution in the geometrical-optics limit
- Kinetic effects. Collisionless damping and instabilities

National Undergraduate Fellowship Program
Introductory Course in Plasma Physics

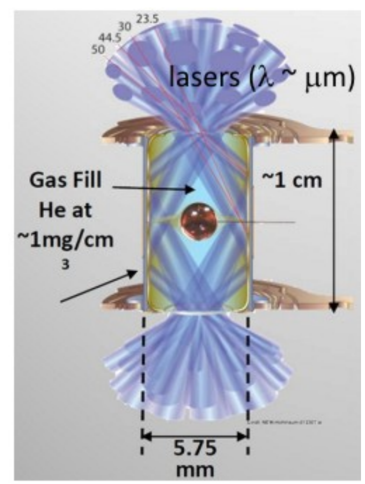
Princeton Plasma Physics Laboratory

Why is it important to understand waves in plasmas?

- Waves are naturally found in plasmas and have to be dealt with.
  This includes instabilities, fluctuations, wave-induced transport. . .
- Waves can deliver and/or manipulate energy-momentum in plasma
  - heating, current drive, particle acceleration
  - mode stabilization, α channeling
  - Raman and Brillouin scattering
- Plasma diagnostics: interferometry, Faraday rotation, Thomson scattering. . .



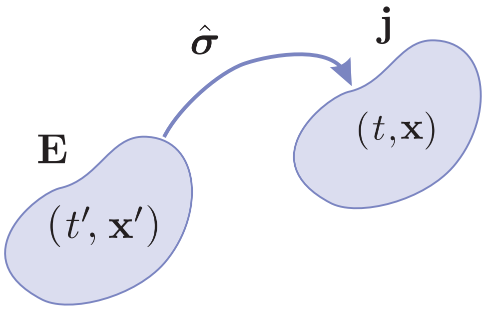
gyrotron (λ ~ mm)



Temporal and spatial dispersion

- Use Maxwell's equations to derive a wave equation for $\mathbf{E}$:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$



$$\boxed{\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t}}$$

- In weak fields, one can anticipate $\mathbf{j}$ to be linear in $\mathbf{E}$
  - Nondispersive medium: $\mathbf{j}(t, \mathbf{x}) = \hat{\boldsymbol{\sigma}}(t, \mathbf{x}) \cdot \mathbf{E}(t, \mathbf{x})$
  - Dispersive medium: *nonlocal* connection between $\mathbf{j}$ and $\mathbf{E}$

$$\boxed{\mathbf{j}(t, \mathbf{x}) = \int \hat{\boldsymbol{\sigma}}(t - t', \mathbf{x} - \mathbf{x}') \cdot \mathbf{E}(t', \mathbf{x}')\, dt'\, d^3x'}$$

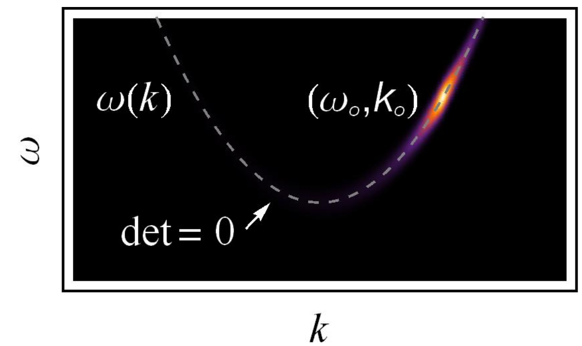
Dispersion relation, $\omega = \omega(\mathrm{k})$. Phase velocity and group velocity

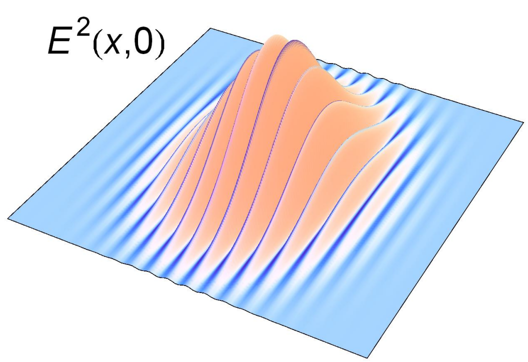
- The Fourier transform gives* $\mathbf{j}_{\omega,\mathbf{k}} = \hat{\boldsymbol{\sigma}}(\omega, \mathbf{k}) \cdot \mathbf{E}_{\omega,\mathbf{k}}, \quad \nabla \to i\mathbf{k}, \quad \partial_t \to -i\omega$

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_{\omega,\mathbf{k}}) + \frac{\omega^2}{c^2}\mathbf{E}_{\omega,\mathbf{k}} = -\frac{4\pi i\omega}{c^2}\hat{\boldsymbol{\sigma}}(\omega, \mathbf{k}) \cdot \mathbf{E}_{\omega,\mathbf{k}}$$

$$\left[\mathbf{k}\mathbf{k} - k^2\hat{\mathbf{1}} + \frac{\omega^2}{c^2}\hat{\boldsymbol{\epsilon}}(\omega, k)\right] \cdot \mathbf{E}_{\omega,\mathbf{k}} = 0$$

$\hat{\boldsymbol{\epsilon}} \equiv \hat{\mathbf{1}} + 4\pi i\hat{\boldsymbol{\sigma}}/\omega$ is the dielectric tensor





$$E(t, x) = \int \mathcal{E}(k)\, e^{-i\omega(k)t + ikx}\, dk$$

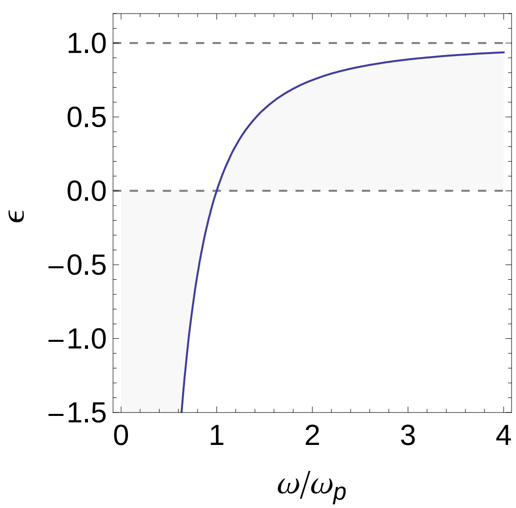
$$\omega(k) \approx \underbrace{\omega(k_0)}_{\omega_0} + \underbrace{\omega'(k_0)}_{v_g}\Delta k, \quad v_p = \omega_0/k_0$$

$$E(t, x) = e^{ik_0(x - v_p t)} \times \text{Amplitude}_0(x - v_g t)$$

*Strictly speaking, this is not precisely true if there is damping or instabilities

Dielectric tensor of cold nonmagnetized plasma

- Need to find $\hat{\boldsymbol{\sigma}}(\omega, \mathbf{k})$ from $\mathbf{j} = \hat{\boldsymbol{\sigma}}(\omega, \mathbf{k}) \cdot \mathbf{E} \sim e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$, using that $\mathbf{j}_e = e n_e \mathbf{v}_e$



$$\underbrace{\frac{\partial \mathbf{v}_e}{\partial t}}_{-i\omega \mathbf{v}_e} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e = \frac{e}{m_e}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_e \times \mathbf{B}\right)$$

$$\mathbf{v}_e \approx ie\mathbf{E}/m_e\omega$$

$$\mathbf{j} = e(n_0 + \delta n_e)\mathbf{v}_e \approx e n_0 \mathbf{v}_e = \underbrace{\frac{i n_0 e^2}{m_e \omega}}_{\hat{\boldsymbol{\sigma}}} \mathbf{E}$$

- $\hat{\boldsymbol{\epsilon}} \equiv \hat{\mathbf{1}} + 4\pi i \hat{\boldsymbol{\sigma}}/\omega$ is diagonal and independent of $\mathbf{k}$

$$\hat{\boldsymbol{\epsilon}} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$$

  - $\epsilon = 1 - \omega_p^2/\omega^2$ is the dielectric function
  - $\omega_p = \sqrt{4\pi n_0 e^2/m_e}$ is the “plasma frequency”

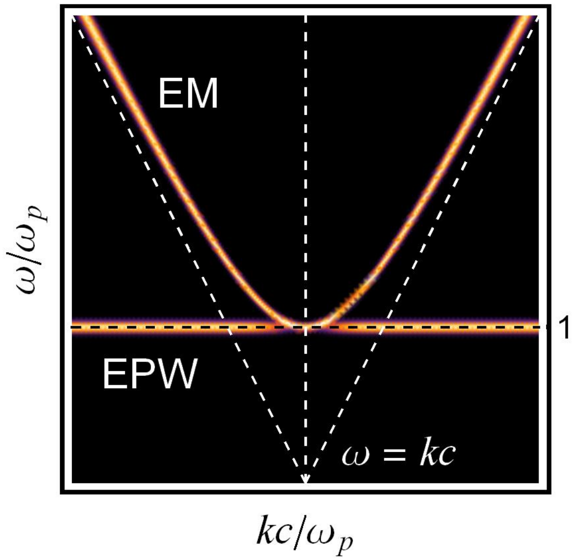
Electron waves in cold nonmagnetized plasma, $\epsilon = 1 - \omega_p^2/\omega^2$

- Transverse electromagnetic waves, $\mathbf{k} \perp \mathbf{E}$

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E})^{\to 0} - k^2\mathbf{E} + \frac{\omega^2}{c^2}\epsilon\mathbf{E} = 0$$

$$\omega^2 = \omega_p^2 + k^2c^2$$

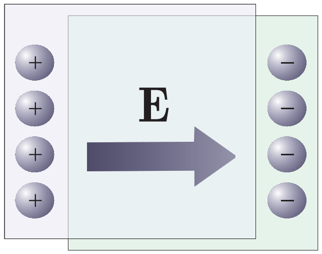
$$\mathbf{v}_p = \omega/k > c, \quad \mathbf{v}_g = c^2/v_p < c$$



- Electron plasma waves, or Langmuir oscillations, $\mathbf{k} \parallel \mathbf{E}$

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E})^{\to 0} + \frac{\omega^2}{c^2}\epsilon\mathbf{E} = 0$$

$$\omega^2 = \omega_p^2, \quad \mathbf{v}_p = \omega_p/k, \quad \mathbf{v}_g = 0$$



Waves in warm nonmagnetized plasma

- Now include the electron pressure, $p_e = p_{e0}(n_e/n_0)^\gamma$, and also ions

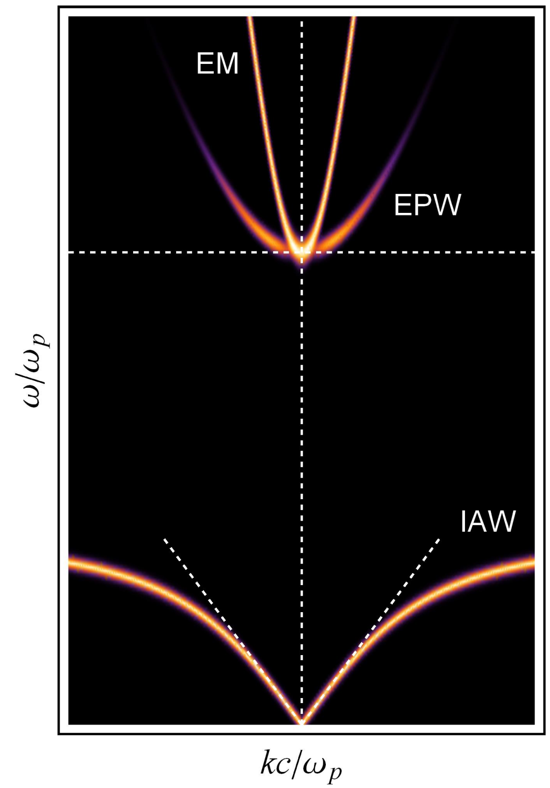
$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = \frac{e_s}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_s \times \mathbf{B}\right) - \frac{\nabla p_s}{n_s m_s}, \qquad \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0$$

  - EM waves are almost unaffected, $\omega^2 \approx \omega_p^2 + k^2c^2$
  - Electron plasma waves (EPW) attain nonzero $\mathbf{v}_g$

$$\boxed{\omega^2 = \omega_p^2 + 3k^2 v_{Te}^2, \quad v_{Te}^2 = T_e/m_e}$$

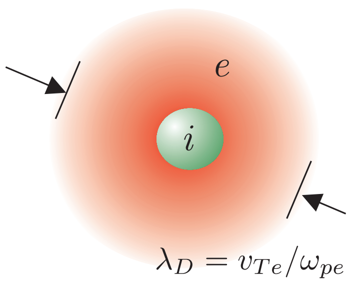
  - New branch – ion acoustic waves (IAW)

$$\boxed{\omega^2 \approx k^2 C_s^2, \quad C_s^2 \approx ZT_e/m_i}$$



Ion acoustic waves in detail

- Electrostatic waves determined by the ion inertia and electron pressure



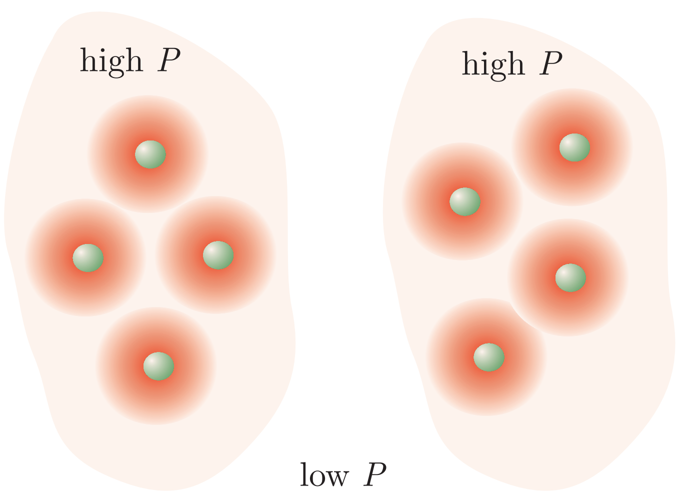
  - Debye shielding of a test charge:

$$n_e = n_0 \exp(e\phi/T_e) \approx n_{0e}(1 + e\phi/T_e)$$

  - Ions are followed by electrons; those carry pressure P_e

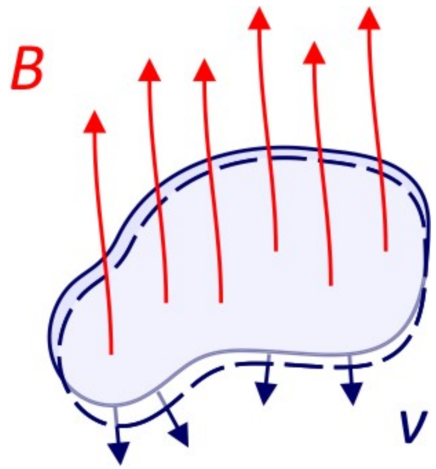
- Dispersion relation (cf. $1 - \omega_{pe}^2/\omega^2 = 0$)

$$1 - \underbrace{\frac{\omega_{pi}^2}{\omega^2}}_{\text{ions}} + \underbrace{\frac{1}{k^2\lambda_D^2}}_{\text{electrons}} = 0$$



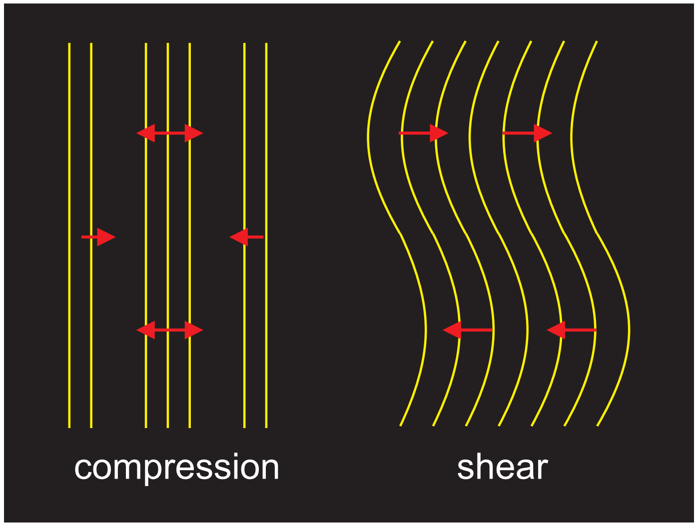
- Is it possible to have low-frequency waves at *negligible* T_e?

Frozen-in law. Magnetic fields can ring!



- Faraday's "frozen-in law" for collisionless plasma:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \text{const}$$



  - Plasma motion causes field distortion
  - Perturbed $\mathbf{B}$ creates a restoring force

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = \underbrace{\frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{4\pi} - \frac{\nabla \mathbf{B}^2}{8\pi}}_{\mathbf{j} \times \mathbf{B}/c}$$

Magnetohydrodynamic (MHD) waves, $\omega \ll \Omega_i$

$$\frac{m_e}{c}\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right]^{\to 0} = \mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E}$$

$$V_A \equiv \frac{B_0}{\sqrt{4\pi\rho_0}}$$ – Alfvén speed

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})$$

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi} - \frac{\nabla \mathbf{B}^2}{8\pi}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \qquad \rho \approx m_i n$$

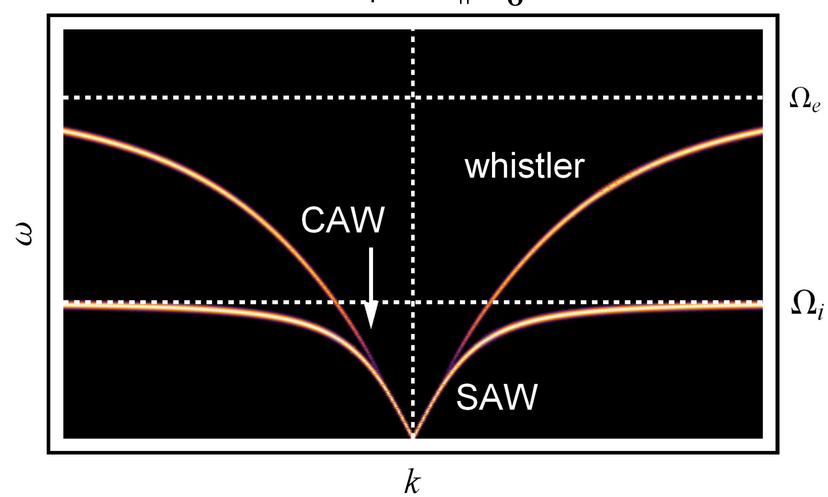
- Compressional Alfvén wave:

$$\omega^2 \approx k^2 V_A^2, \quad \mathbf{v}_g \parallel \mathbf{k}$$

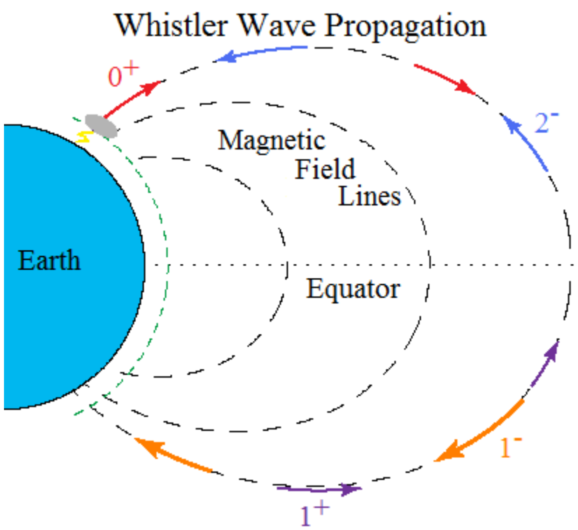
- Shear Alfvén wave, $\nabla\cdot\mathbf{v} = 0$

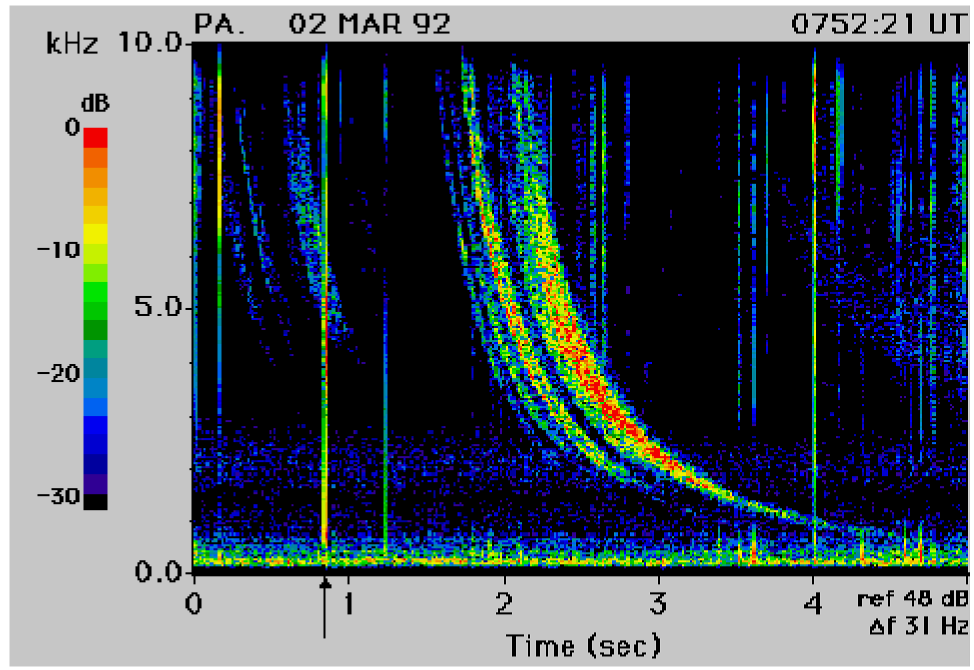
$$\omega^2 \approx k_\parallel^2 V_A^2, \quad \mathbf{v}_g \parallel \mathbf{B}_0$$

example: $\mathbf{k} \parallel \mathbf{B_0}$



- Whistlers: R-polarized EM waves propagating with $\mathbf{v}_g \parallel \mathbf{B}_0$ at $\omega < \Omega_e$





- Earth: triggered by lightnings, $\omega \sim 10^3 - 10^4$ Hz (audio range!), $v_g \sim \sqrt{\omega}$
  - ↳ Signals with lower ω arrive at a detector *later*. Sounds like whistle

Dielectric tensor of cold magnetized plasma

- Let us calculate $\hat{\boldsymbol{\sigma}}(\omega, \mathbf{k})$ from $\mathbf{j} = \hat{\boldsymbol{\sigma}}(\omega, \mathbf{k}) \cdot \mathbf{E} \sim e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$, using that

$$\mathbf{j} = \sum_s e_s (n_{0s} + \delta n_s) \mathbf{v}_s \approx \sum_s e_s n_{0s} \mathbf{v}_s$$

$$-i\omega \mathbf{v}_s \approx \frac{e_s}{m_s} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_s \times \mathbf{B}_0 \right)$$

- Dielectric tensor $\hat{\boldsymbol{\epsilon}} \equiv 1 + 4\pi i \hat{\boldsymbol{\sigma}}/\omega$:

$$\hat{\boldsymbol{\epsilon}} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$$S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}$$

$$D = \sum_s \frac{\Omega_s}{\omega} \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}$$

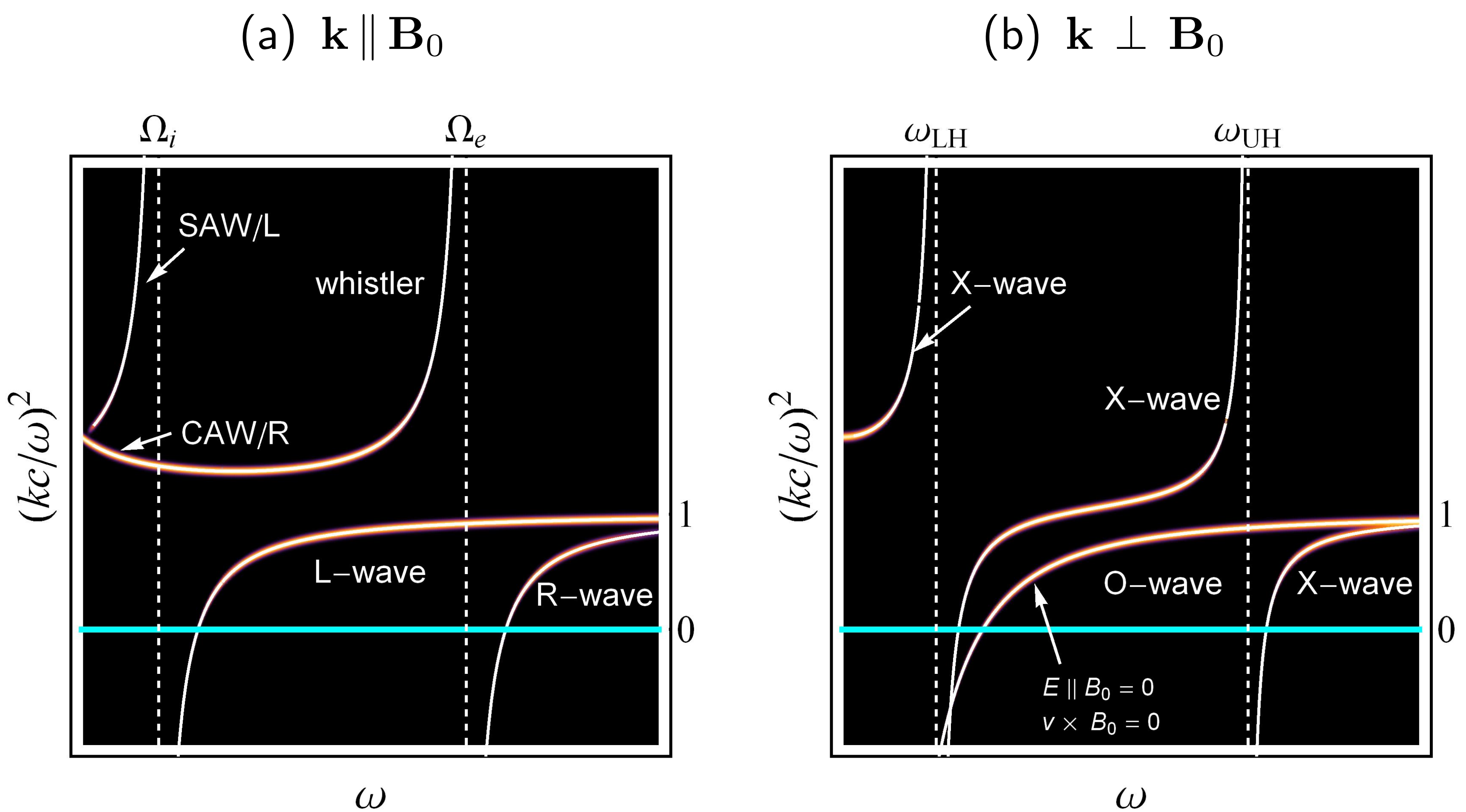
$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

$$\omega_{ps}^2 = 4\pi n_{0s} e_s^2 / m_s$$

  - Hermitian tensor. Within this model, waves do not dissipate
  - Singular at cyclotron resonances, $\omega = \Omega_s$, where $\Omega_s = e_s B_0 / m_s c$

Waves in cold magnetized plasma. Summary

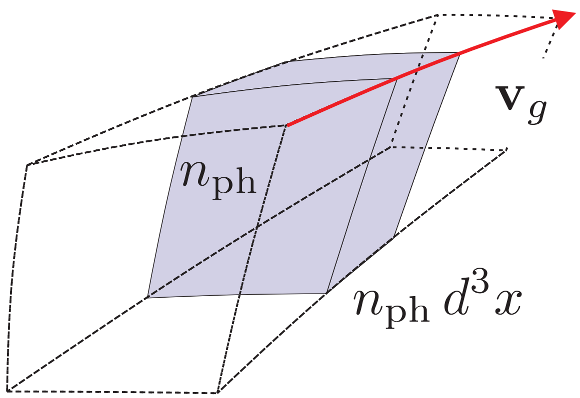
- Electron-ion plasma, one type of ions, gyrofrequencies $\Omega_s = |e_s| B_0 / m_s c$



Dynamics in the geometrical-optics limit. Action conservation

- In nonstationary inhomogeneous plasma $\hat{\boldsymbol{\sigma}}(\omega, \mathbf{k})$ will depend also on $(t, \mathbf{x})$
- From QM: each wave can be quantized into photons, “plasmons”, etc.

$$\hbar \int n_{\text{ph}}\, d^3x = \int \frac{\mathcal{E}}{\omega}\, d^3x = \text{const}$$



Phys. Rev. A 86, 053834 (2012); Phys. Lett. A 378, 1598 (2014)

$$n_{\text{ph}}\hbar = \mathcal{E}/\omega, \quad \mathcal{E} = n_{\text{ph}}\hbar\omega, \quad \mathbf{P} = n_{\text{ph}}\hbar\mathbf{k}$$

$$\frac{d\mathbf{x}}{dt} = \frac{\partial(\hbar\omega)}{\partial(\hbar\mathbf{k})} \equiv \mathbf{v}_g, \qquad \frac{d(\hbar\mathbf{k})}{dt} = -\frac{\partial(\hbar\omega)}{\partial\mathbf{x}}$$

$$\partial_t n_{\text{ph}} + \nabla\cdot(\mathbf{v}_g n_{\text{ph}}) = 0 \qquad \ldots \times \hbar$$

$$\partial_t(\mathcal{E}/\omega) + \nabla\cdot(\mathbf{v}_g\mathcal{E}/\omega) = 0$$

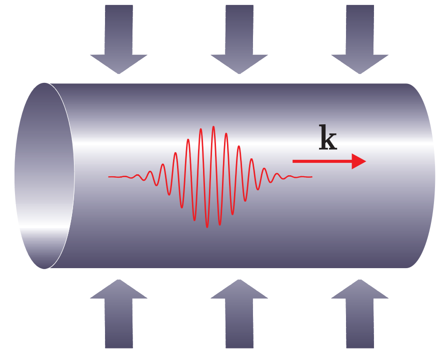
- The “ray equations” and the “action density”, $\mathcal{E}/\omega$, are classical (no $\hbar$)
- Applicability conditions (GO): plasma parameters vary slowly ($\partial_t \ll \omega$, $\nabla \ll k$)

Action conservation theorem: applications

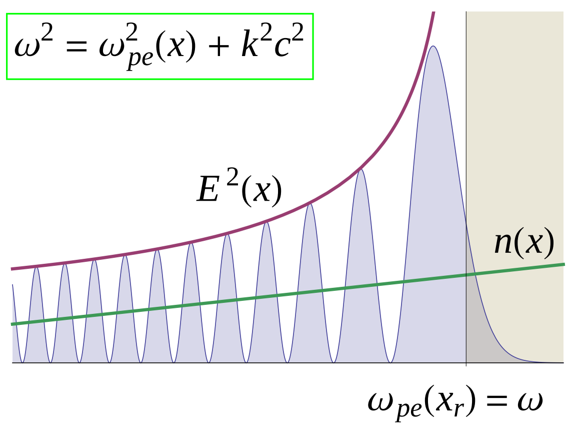
- Example: transverse EM wave in nonmagnetized plasma, $\mathcal{E} = |E|^2/8\pi$
  - Homogeneous nonstationary plasma:

$$\omega^2 = \omega_p^2(t) + k^2c^2, \quad \dot{k} = -\partial_x \omega \equiv 0$$

$$\frac{\int \mathcal{E}_2\, d^3x}{\omega_2} = \frac{\int \mathcal{E}_1\, d^3x}{\omega_1}$$



  - Inhomogeneous stationary plasma (one-dimensional problem):

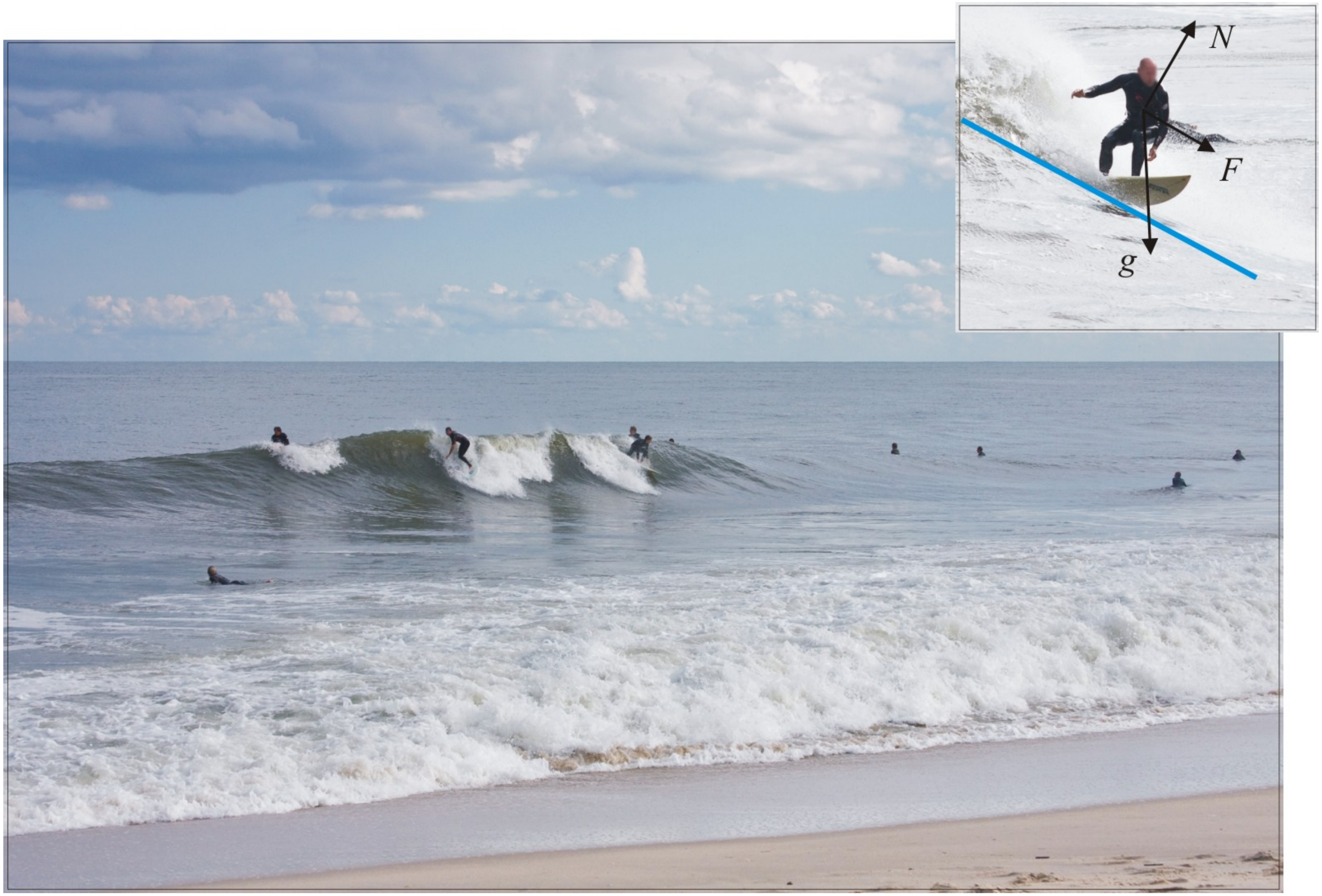


$$\partial_t(\mathcal{E}/\omega)^{0} + \partial_x(v_g \mathcal{E}/\omega) = 0$$

$$v_g = \partial_k \omega(x,k), \quad \dot{\omega} = \partial_t \omega \equiv 0$$

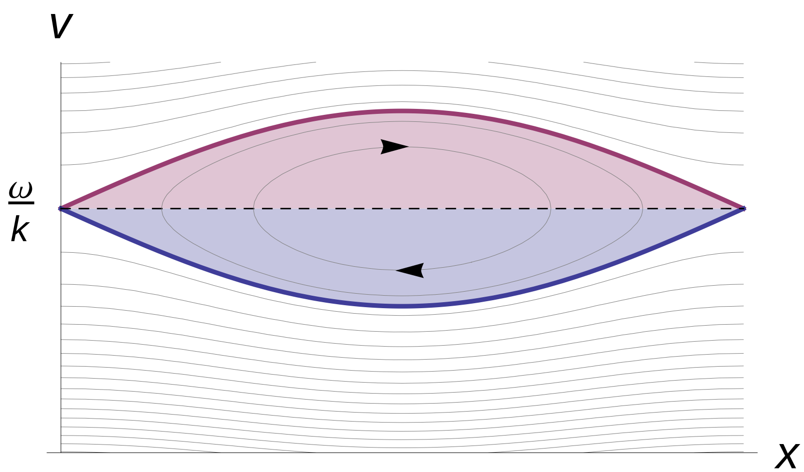
$$E \sim k^{-1/2}(x) \sim \left[1 - \omega_p^2(x)/\omega^2\right]^{-1/4}$$

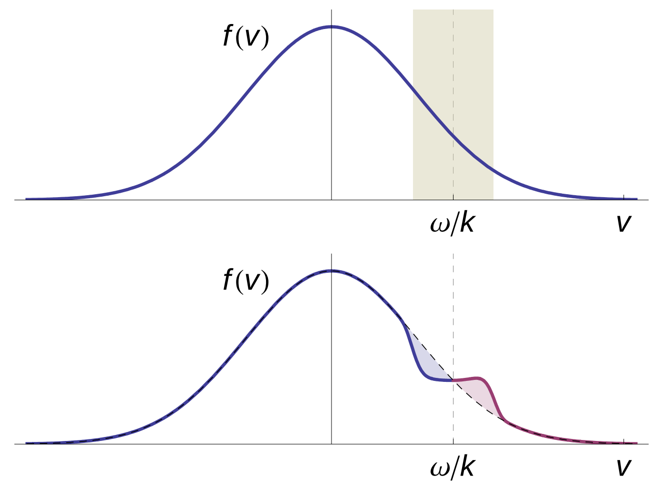
Resonant interaction with particles occurs when $v \approx \omega/k$

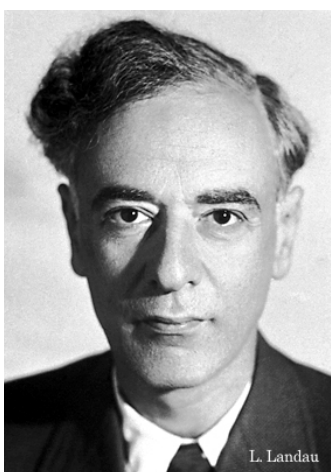


Resonant interaction with particles. Landau damping

- In the wave rest frame, $v' = v'(x')$. The perturbations are strongest at $v \approx \omega/k$



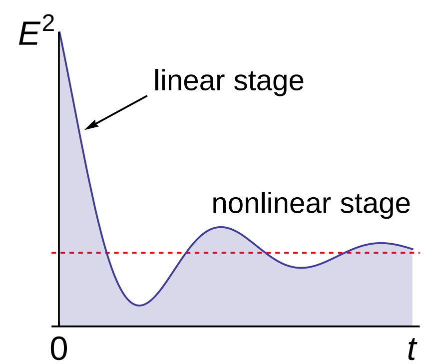




- Linear stage: $E \sim e^{-\gamma t}$. For Langmuir waves,

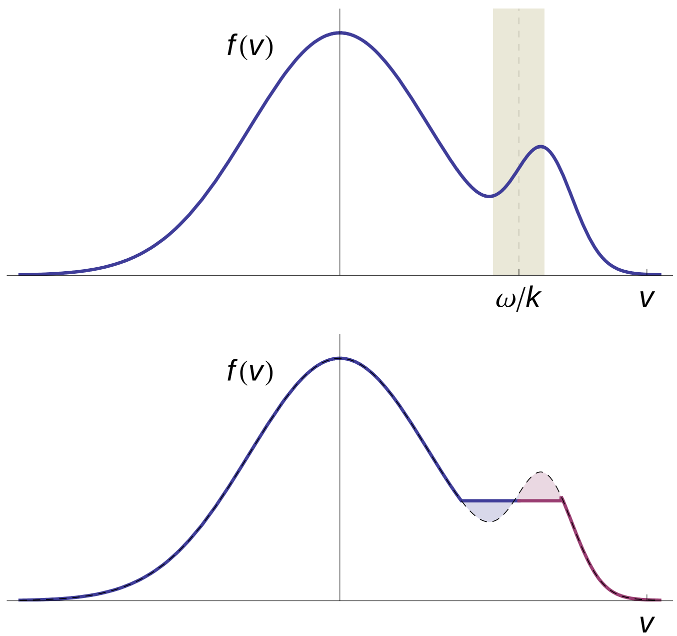
$$\gamma \approx \frac{\omega_{pe}}{\kappa^3}\sqrt{\frac{\pi}{8}}\exp\left(-\frac{1}{2\kappa^2}-\frac{3}{2}\right)$$

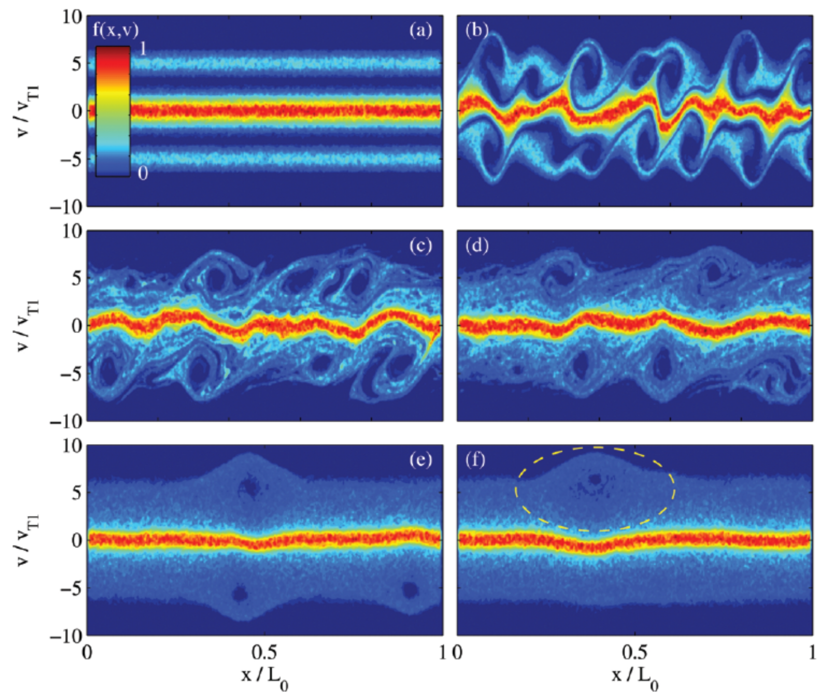
$$\kappa \equiv k v_{Te}/\omega_{pe} \ \sim \ v_{Te}/v_p \lesssim 1$$



Bump-on-tail instability as a paradigmatic kinetic instability

- If $f'(\omega/k) > 0$, particles, on average, are *decelerated*, so the wave is unstable





  - Like an inverted medium. A spectrum of waves is generated, flattening $f(v)$
  - The saturated state can be an essentially nonlinear wave, $\omega(k, a)$

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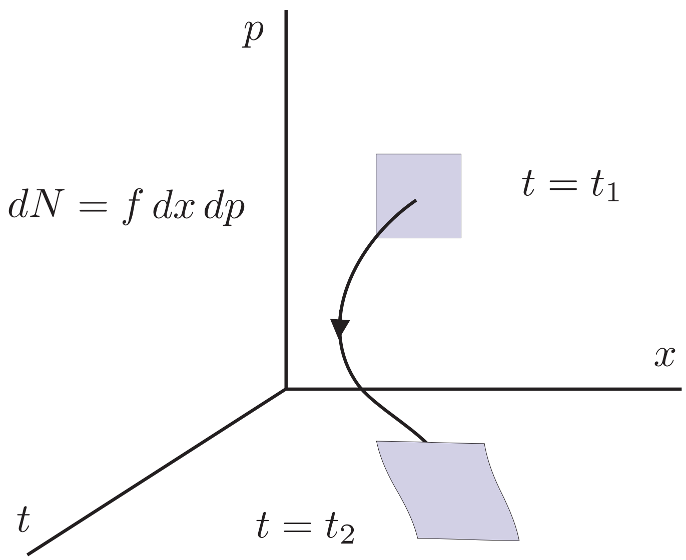
Kinetic treatment. Vlasov equation

- Our fluid equations do not resolve kinetic effects. A general theory is needed



- dN = const in $d\text{V} = d^3x\, d^3p$
- Liouville's theorem: $d\text{V}$ = const
- Thus, $f \equiv dN/d\text{V}$ = const, i.e.,

$$\frac{d}{dt} f(t, \mathbf{x}(t), \mathbf{p}(t)) = 0$$

- Vlasov equation:

$$\dot{\mathbf{x}}(t) = \mathbf{v}(t)$$

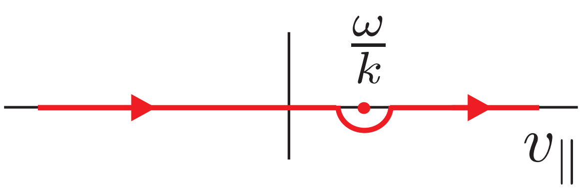
$$\dot{\mathbf{p}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{p}(t), t)$$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

Dielectric tensor of hot nonmagnetized plasma ($\mathbf{v}\times\mathbf{B}/c$ is neglected)

- Let us calculate $\hat{\boldsymbol{\sigma}}(\omega,\mathbf{k})$ from $\mathbf{j}=\hat{\boldsymbol{\sigma}}(\omega,\mathbf{k})\cdot\mathbf{E}\ \sim\ e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}$, using that

$$j_m=\sum_s e_s\int v_m f_s\,d^3p=\sum_s e_s\int v_m(f_{0s}+\tilde{f}_s)\,d^3p=\sum_s e_s\int v_m\tilde{f}_s\,d^3p$$



$$\underbrace{\frac{\partial f_s}{\partial t}+\mathbf{v}\cdot\nabla f_s}_{-i(\omega-\mathbf{k}\cdot\mathbf{v})\tilde{f}_s}+e_sE_\ell\left(\frac{\partial f_{s0}}{\partial p_\ell}+\cancel{\frac{\partial\tilde{f}_s}{\partial p_\ell}}\right)=0$$

$$\hat{\boldsymbol{\epsilon}}=\begin{pmatrix}\epsilon_\perp & 0 & 0\\ 0 & \epsilon_\perp & 0\\ 0 & 0 & \epsilon_\parallel\end{pmatrix}$$

$$\boxed{\epsilon_{m\ell}=\delta_{m\ell}+\sum_s\frac{\omega_{ps}^2}{\omega}\int\frac{v_m}{\omega-\mathbf{k}\cdot\mathbf{v}}\frac{\partial f_{0s}}{\partial v_\ell}\,d^3p}$$

- The pole is encircled in the complex-$v_\parallel$ plane. Thus $\hat{\boldsymbol{\epsilon}}$ is *not* Hermitian

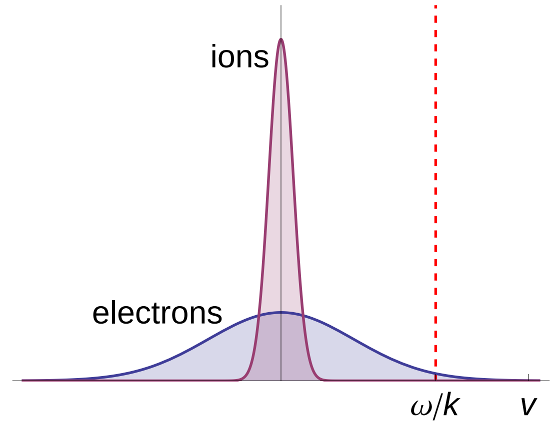
  ↳ Collisionless damping and instabilities

Electrostatic waves in hot magnetized plasma

$$\underbrace{\mathbf{k}\times(\mathbf{k}\times\mathbf{E})}_{\to 0} + \frac{\omega^2}{c^2}\underbrace{\hat{\epsilon}\cdot\mathbf{E}}_{\epsilon_\parallel \mathbf{E}} = 0, \qquad \epsilon_\parallel = 0$$

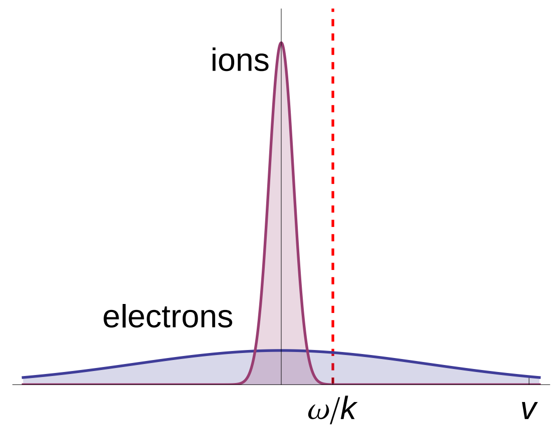
- EPW: warm electrons, cold ions

$$\epsilon_\parallel \approx 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2 v_{Te}^2}{\omega^2}\right) + i\sqrt{\frac{\pi}{2}}\,\frac{\omega}{\omega_p}\,(k\lambda_D)^{-3}\exp\left(-\frac{v_p^2}{2v_{Te}^2}\right)$$



- Ion sound: hot electrons, cold ions

$$\epsilon_\parallel \approx 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{1}{k^2\lambda_D^2}\left(1 + i\sqrt{\frac{\pi}{2}}\,\frac{v_p}{v_{Te}}\right)$$



Example: dielectric tensor of hot magnetized plasma

$$\epsilon(\omega,\mathbf{k}) = \mathbf{1} + \sum_s \chi_s(\omega,\mathbf{k})$$

$$\chi_s = \frac{\omega_{p0,s}^2}{\omega\Omega_{0,s}} \sum_{n=-\infty}^{\infty} \int_0^{\infty} 2\pi p_\perp \, dp_\perp \int_{-\infty}^{\infty} dp_\parallel \left(\frac{\Omega}{\omega - k_\parallel v_\parallel - n\Omega} \mathbf{S}_n \right)_s$$

$$\mathbf{S}_n = \begin{pmatrix} \frac{n^2 J_n^2}{z^2} p_\perp U & \frac{inJ_nJ'_n}{z} p_\perp U & \frac{nJ_n^2}{z} p_\perp W \\ -\frac{inJ_nJ'_n}{z} p_\perp U & (J'_n)^2 p_\perp U & -iJ_nJ'_n p_\perp W \\ \frac{nJ_n^2}{z} p_\parallel U & iJ_nJ'_n p_\parallel U & J_n^2 p_\parallel W \end{pmatrix}$$

$$U = \frac{\partial f_0}{\partial p_\perp} + \frac{k_\parallel}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp} \right),$$

$$V = \frac{k_\perp}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp} \right),$$

$$W = \left(1 - \frac{n\Omega}{\omega} \right) \frac{\partial f_0}{\partial p_\parallel} + \frac{n\Omega p_\parallel}{\omega p_\perp} \frac{\partial f_0}{\partial p_\perp}.$$

$J_n(z)$, z denotes $k_\perp v_\perp / \Omega$

Summary

- Plasma has many degrees of freedom, so it supports many waves, e.g.,
  - Electromagnetic waves, similar to those in vacuum if $\omega \gg \omega_p$
  - Langmuir waves, or electrostatic plasma oscillations, $\omega \approx \omega_p$
  - Ion acoustic waves, or ion sound, $\omega \approx C_s k$
  - Alfvén waves, or low-frequency B-field oscillations,
  - Whistlers. . .
- Quasimonochromatic waves are described by geometrical-optics equations
  - Rays travel as particles with energy $\hbar\omega(t, \mathbf{x}, \hbar\mathbf{k})$ and momentum $\hbar\mathbf{k}$
  - These "quasiparticles" are conserved, so $\int(\mathcal{E}/\omega)d^3x = \text{const}$
- Resonant interactions with particles appear as damping/instabilities
- The physics of incoherent and nonlinear waves is a whole other story

Some books on waves in plasmas and basic wave physics

[*Aleksandrov* et al.*, 1984*]
A. F. Aleksandrov and L. S. Bogdankevich and A. A. Rukhadze, *Principles of plasma electrodynamics* (Springer-Verlag, New York, 1984).

[*Brambilla, 1998*]
M. Brambilla, *Kinetic theory of plasma waves: homogeneous plasmas* (Clarendon Press, Oxford, 1998).

[*Krall and Trivelpiece, 1973*]
N. A. Krall and A. W. Trivelpiece, *Principles of plasma physics* (McGraw-Hill, New York, 1973).

[*Pécseli, 2013*]
H. L. Pécseli, *Waves and oscillations in plasmas* (CRC Press, Boca Raton, 2013).

[*Stix, 1992*]
T. H. Stix, *Waves in plasmas* (AIP, New York, 1992).

[*Tracy* et al.*, 2014*]
E. R. Tracy and A. J. Brizard and A. S. Richardson and A. N. Kaufman, *Ray tracing and beyond: phase space methods in plasma wave theory* (Cambidge University Press, New York, 2014).

[*Whitham, 1974*]
G. B. Whitham, *Linear and nonlinear waves* (Wiley, New York, 1974).